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On the effect of torsion on a helical pipe flow

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An orthogonal coordinate system along a generic spatial curve has been introduced, and the Navier–Stokes equations for a steady incompressible viscous flow have been explicitly written in this frame of reference. As an application the flow in a helical pipe has been studied, and, for radii of curvature and torsion small compared with the radius of the pipe, the flow has been considered as a perturbed Poiseuille flow. The result is that for curvatures and torsions of the same order and for low Reynolds number the curvature induces on the flow a first-order effect on the parameter $\epsilon = \kappa a$, where κ is the curvature and a the radius of the pipe, while the effect of the torsion on the flow is of the second order in ϵ. This last result disagrees with those of Wang (1981), who, adopting a non-orthogonal coordinate system, found a first-order effect of torsion on the flow.

1. Introduction

The effect of torsion on the flow in helical pipes is not fully understood. Particularly interesting from a theoretical point of view is to determine whether the torsion has an effect comparable to the curvature in developing secondary circulating flows in a plane normal to the axis of the pipe.

Wang (1981) in a recent article has studied this problem by introducing a non orthogonal, helical coordinate system. In terms of the parameter $\epsilon = \kappa a \ll 1$, where κ is the curvature and a the radius of the pipe, and for curvatures and torsions of the same order, his results give a first-order effect in ϵ of the torsion on the secondary flow, comparable to the effect of the curvature and perturbative of the primary flow, the Poiseuille flow. For zero torsion he recovers the two symmetric cells found by Dean (1927), and he states that the effect of the first-order term due to the torsion depends on the Reynolds number, and is in some cases so dominant that the two recirculating cells become a single one.

In our work we introduce an orthogonal system of coordinates along a spatial curve, and we derive the Navier–Stokes equation in this metric by using simple scale factors. As a first application we attempt to test the effect of the torsion on a helical pipe flow, for constant values of curvature and torsion. In this analysis we adopt the same approximation used by Dean (1927), and we consider the motion in the helical pipe as a perturbation of the main Poiseuille flow. Our perturbative parameter is ϵ, and the two other parameters $\lambda = \tau/\kappa$, where τ is the torsion of the helix, and the Reynolds number $\mathscr{R}$ of the main flow in the pipe are considered to be of order unity.

2. An orthogonal coordinate system following a spatial curve. The metric and the Navier–Stokes equations

Let us consider a spatial curve, with s the arc length, $\mathbf{R}(s)$ the curve position, $\mathbf{T}$ the tangent, and $\mathbf{N}$ and $\mathbf{B}$ the normal and the binormal.

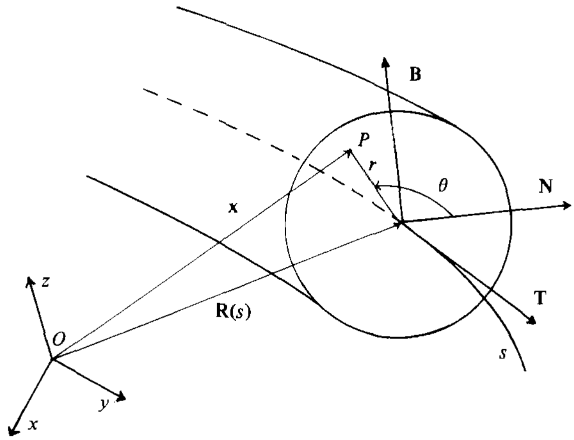

FIGURE 1. Wang's (1981) coordinate system.

Wang (1981) constructed a coordinate system (s, r, θ) such that any Cartesian position vector $\mathbf{x}$ can be expressed as (see figure 1)

$$\mathbf{x} = P - O = \mathbf{R}(s) + r\cos\theta\mathbf{N}(s) + r\sin\theta\mathbf{B}(s). \tag{1}$$

Using the relations

$$\mathbf{T} = \frac{d\mathbf{R}}{ds}, \quad \mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds}, \quad \mathbf{B} = \mathbf{T}\times\mathbf{N}, \tag{2a, b, c}$$

and the Frenet formulae

$$\frac{d\mathbf{N}}{ds} = \tau\mathbf{B} - \kappa\mathbf{T}, \quad \frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}, \tag{3a, b}$$

where κ and τ are the curvature and the torsion, Wang (1981) obtains the metric

$$d\mathbf{x}\,.\,d\mathbf{x} = [(1-\kappa r\cos\theta)^2 + \tau^2 r^2](ds)^2 + (dr)^2 + r^2(d\theta)^2 + 2\tau r^2\,ds\,d\theta. \tag{4}$$

The last term in (4) indicates that the coordinate system (s, r, θ) is non-orthogonal, and tensor analysis is necessary to derive the governing equations.

Now we will show that it is possible to obtain a new orthogonal coordinate system. Making use of the fact that the origin of the angle θ in the plane normal to the axis is arbitrary, we can express the Cartesian vector $\mathbf{x}$ as (see figure 2)

$$\mathbf{x} = P - O = \mathbf{R}(s) + r\cos(\theta + \phi(s) + \phi_0)\,\mathbf{N}(s) + r\sin(\theta + \phi(s) + \phi_0)\,\mathbf{B}(s), \tag{5}$$

where

$$\phi(s) = -\int_{s_0}^{s} \tau(s')\,ds', \tag{6}$$

and with ϕ_0 and s_0 taking arbitrary values.

We choose for ϕ_0 the particular value $\frac{1}{2}\pi$, and we can write for the change in the position vector $d\mathbf{x}$ corresponding to increments in the coordinates

$$d\mathbf{x} = ds(1 + \kappa r\sin(\theta + \phi))\,\mathbf{a}_s + dr\,\mathbf{a}_r + rd\theta\,\mathbf{a}_\theta, \tag{7}$$

where the unit vectors parallel to the coordinate lines are

$$\mathbf{a}_s = \mathbf{T}, \tag{8a}$$

$$\mathbf{a}_r = \mathbf{B}\cos(\theta + \phi) - \mathbf{N}\sin(\theta + \phi), \tag{8b}$$

$$\mathbf{a}_\theta = -\mathbf{B}\sin(\theta + \phi) - \mathbf{N}\cos(\theta + \phi), \tag{8c}$$

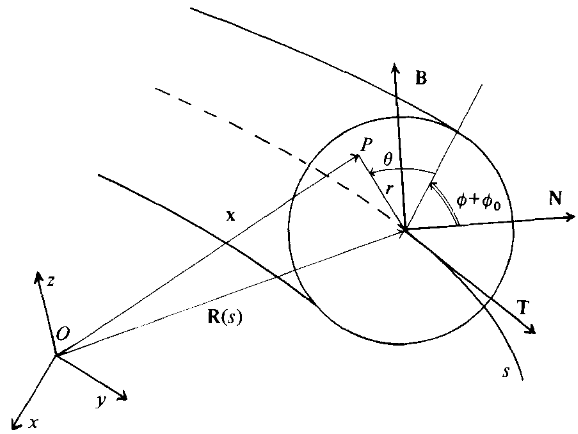

FIGURE 2. Present coordinate system.

and finally we obtain the following metric for our new coordinate system:

$$d\mathbf{x}\,.\,d\mathbf{x} = [1+\kappa r\sin(\theta+\phi)]^2\,(ds)^2+(dr)^2+r^2(d\theta)^2, \tag{9}$$

which is clearly an orthogonal one.

The choice of $\phi_0 = \frac{1}{2}\pi$ is such that when $\tau = 0$ we obtain the system of reference ordinarily used in the study of the flow in a curved pipe (see e.g. Ward-Smith 1980, p. 249):

$$d\mathbf{x}\,.\,d\mathbf{x} = [1+\kappa r\sin\theta]^2\,(ds)^2+(dr)^2+r^2(d\theta)^2. \tag{10}$$

Let us now write the Navier–Stokes equations for an incompressible steady viscous fluid in the coordinates s, r, θ with the orthogonal metric (9). In the following we will write κ, τ and ϕ, but it is intended that generally they are functions of the arc length s.

We can write for the velocity vector $\mathbf{v}$

$$\nabla\,.\,\mathbf{v} = 0, \quad (\mathbf{v}\,.\,\nabla)\,\mathbf{v} = -\nabla p - \nu\nabla\times\nabla\times\mathbf{v}, \tag{11a, b}$$

where p and ν are the kinematic pressure and the kinematic viscosity. In orthogonal curvilinear coordinate system the expression for the divergence is

$$\frac{1}{h_1h_2h_3}\left[\frac{\partial(h_2h_3v_1)}{\partial\xi_1}+\frac{\partial(h_3h_1v_2)}{\partial\xi_2}+\frac{\partial(h_1h_2v_3)}{\partial\xi_3}\right], \tag{12}$$

where ξ_1, ξ_2, ξ_3 are the orthogonal coordinates, h_1, h_2, h_3 the scale factors, and v_1, v_2, v_3 the components of the velocity. The components of the gradient are given by

$$\frac{1}{h_1}\frac{\partial}{\partial\xi_1}, \quad \frac{1}{h_2}\frac{\partial}{\partial\xi_2}, \quad \frac{1}{h_3}\frac{\partial}{\partial\xi_3} \tag{13}$$

(see Batchelor 1967), and the components of $\boldsymbol{\omega} = \nabla\times\mathbf{v}$ are

$$\omega_1 = \frac{1}{h_2h_3}\left(\frac{\partial}{\partial\xi_2}(h_3v_3)-\frac{\partial}{\partial\xi_3}(h_2v_2)\right), \quad \omega_2 = \frac{1}{h_3h_1}\left(\frac{\partial}{\partial\xi_3}(h_1v_1)-\frac{\partial}{\partial\xi_1}(h_3v_3)\right), \tag{14a, b}$$

$$\omega_3 = \frac{1}{h_1h_2}\left(\frac{\partial}{\partial\xi_1}(h_2v_2)-\frac{\partial}{\partial\xi_2}(h_1v_1)\right), \tag{14c}$$

while the components of $(\mathbf{v}\,.\,\nabla)\mathbf{v}$ are

$$\mathscr{D}v_1+\frac{v_2}{h_2h_1}\left(v_1\frac{\partial h_1}{\partial\xi_2}-v_2\frac{\partial h_2}{\partial\xi_1}\right)+\frac{v_3}{h_3h_1}\left(v_1\frac{\partial h_1}{\partial\xi_3}-v_3\frac{\partial h_3}{\partial\xi_1}\right), \tag{15a}$$

$$\mathscr{D}v_2+\frac{v_3}{h_3h_2}\left(v_2\frac{\partial h_2}{\partial\xi_3}-v_3\frac{\partial h_3}{\partial\xi_2}\right)+\frac{v_1}{h_1h_2}\left(v_2\frac{\partial h_2}{\partial\xi_1}-v_1\frac{\partial h_1}{\partial\xi_2}\right), \tag{15b}$$

$$\mathscr{D}v_3+\frac{v_1}{h_1h_3}\left(v_3\frac{\partial h_3}{\partial\xi_1}-v_1\frac{\partial h_1}{\partial\xi_3}\right)+\frac{v_2}{h_2h_3}\left(v_3\frac{\partial h_3}{\partial\xi_2}-v_2\frac{\partial h_2}{\partial\xi_3}\right), \tag{15c}$$

where the operator $\mathscr{D}$ is given by the expression

$$\mathscr{D}=\frac{v_1}{h_1}\frac{\partial}{\partial\xi_1}+\frac{v_2}{h_2}\frac{\partial}{\partial\xi_2}+\frac{v_3}{h_3}\frac{\partial}{\partial\xi_3}. \tag{16}$$

In our case we have

ξ_i	s	r	θ
h_i	$1+\kappa r\sin(\theta+\phi)$	1	r
v_i	u	v	w

and we obtain the following non-dimensional expressions: for the continuity equation

$$\omega\frac{\partial\tilde{u}}{\partial\tilde{s}}+\frac{\partial\tilde{v}}{\partial\tilde{r}}+\frac{1}{\tilde{r}}\frac{\partial\tilde{w}}{\partial\theta}+\frac{\tilde{v}}{\tilde{r}}+\epsilon\omega\,[\tilde{v}\sin(\theta+\phi)+\tilde{w}\cos(\theta+\phi)]=0; \tag{17}$$

and for the momentum equations

$$\begin{aligned}\tilde{\mathscr{D}}\tilde{u}+\epsilon\omega\tilde{u}[\tilde{v}\sin(\theta+\phi)+\tilde{w}\cos(\theta+\phi)]&=-\omega\frac{\partial\tilde{p}}{\partial\tilde{s}}\\&+\frac{1}{\mathscr{R}}\left[\left(\frac{\partial}{\partial\tilde{r}}+\frac{1}{\tilde{r}}\right)\left(\frac{\partial\tilde{u}}{\partial\tilde{r}}+\epsilon\omega\tilde{u}\sin(\theta+\phi)-\omega\frac{\partial\tilde{v}}{\partial\tilde{s}}\right)\right.\\&\left.+\frac{1}{\tilde{r}}\frac{\partial}{\partial\theta}\left(\frac{1}{\tilde{r}}\frac{\partial\tilde{u}}{\partial\theta}+\epsilon\omega\tilde{u}\cos(\theta+\phi)-\omega\frac{\partial\tilde{w}}{\partial\tilde{s}}\right)\right],\end{aligned} \tag{18a}$$

$$\begin{aligned}\tilde{\mathscr{D}}\tilde{v}-\frac{\tilde{w}^2}{\tilde{r}}-\epsilon\omega\tilde{u}^2\sin(\theta+\phi)&=-\frac{\partial\tilde{p}}{\partial\tilde{r}}\\&-\frac{1}{\mathscr{R}}\left[\left(\frac{1}{\tilde{r}}\frac{\partial}{\partial\theta}+\epsilon\omega\cos(\theta+\phi)\right)\left(\frac{\partial\tilde{w}}{\partial\tilde{r}}+\frac{\tilde{w}}{\tilde{r}}-\frac{1}{\tilde{r}}\frac{\partial\tilde{v}}{\partial\theta}\right)\right.\\&\left.-\omega\frac{\partial}{\partial\tilde{s}}\left(\omega\frac{\partial\tilde{v}}{\partial\tilde{s}}-\frac{\partial\tilde{u}}{\partial\tilde{r}}-\epsilon\omega\tilde{u}\sin(\theta+\phi)\right)\right],\end{aligned} \tag{18b}$$

$$\begin{aligned}\tilde{\mathscr{D}}\tilde{w}+\frac{\tilde{v}\tilde{w}}{\tilde{r}}-\epsilon\omega\tilde{u}^2\cos(\theta+\phi)&=-\frac{1}{\tilde{r}}\frac{\partial\tilde{p}}{\partial\theta}\\&+\frac{1}{\mathscr{R}}\left[\left(\frac{\partial}{\partial\tilde{r}}+\epsilon\omega\sin(\theta+\phi)\right)\left(\frac{\partial\tilde{w}}{\partial\tilde{r}}+\frac{\tilde{w}}{\tilde{r}}-\frac{1}{\tilde{r}}\frac{\partial\tilde{v}}{\partial\theta}\right)\right.\\&\left.-\omega\frac{\partial}{\partial\tilde{s}}\left(\frac{1}{\tilde{r}}\frac{\partial\tilde{u}}{\partial\theta}+\epsilon\omega\tilde{u}\cos(\theta+\phi)-\omega\frac{\partial\tilde{w}}{\partial\tilde{s}}\right)\right].\end{aligned} \tag{18c}$$

Here

$$\omega=\frac{1}{1+\epsilon\tilde{r}\sin(\theta+\phi)},\quad \tilde{\mathscr{D}}=\omega\tilde{u}\frac{\partial}{\partial\tilde{s}}+\tilde{v}\frac{\partial}{\partial\tilde{r}}+\frac{\tilde{w}}{\tilde{r}}\frac{\partial}{\partial\theta}, \tag{19a, b}$$

and
$$\left.\begin{aligned}&\tilde{s}=\frac{s}{a},\quad \tilde{r}=\frac{r}{a},\quad (\tilde{u},\tilde{v},\tilde{w})=\left(\frac{u}{U},\frac{v}{U},\frac{w}{U}\right),\quad \tilde{p}=\frac{p}{U^2},\\ &\epsilon=\kappa a,\quad \lambda=\frac{\tau}{\kappa},\quad \mathscr{R}=\frac{Ua}{\nu},\end{aligned}\right\} \tag{20}$$

where a and U are a reference length and a reference velocity. The validity of this set of equations is for $r \leqslant 1/\kappa$. Otherwise, as Wang (1981) notices, the description of any point in the system is not unique. For κ and τ constant they allow the study of a helical flow, and for $\tau = 0$ and κ constant they reduce to the equations written in toroidal coordinates by Dean (1927).

3. Laminar flow in a helical pipe. First approximation for small curvature and torsion

Let us now apply the equations (17) and (18) to the study of the effect of the torsion in a helical flow. In this case κ and τ are constant, and we adopt a first-approximation method for curvature and torsion small compared with the radius of the pipe, by considering that their effects are a perturbation of the main Poiseuille flow. In this case the reference length a is the radius of the pipe, the reference velocity U is the central velocity of the main Poiseuille flow, and we assume that $\epsilon \ll 1$ and that λ and $\mathscr{R}$ are of order unity. In this case we look for solutions of (17) and (18) in the form
$$\tilde{u}=\tilde{u}_0(\tilde{r})+\epsilon\tilde{u}_1(\theta+\phi,\tilde{r})+\ldots,\quad \tilde{v}=\epsilon\tilde{v}_1(\theta+\phi,\tilde{r})+\ldots, \tag{21a, b}$$
$$\tilde{w}=\epsilon\tilde{w}_1(\theta+\phi,\tilde{r})+\ldots,\quad \tilde{p}=\tilde{p}_0(\tilde{s})+\epsilon\tilde{p}_1(\theta+\phi,\tilde{r})+\ldots, \tag{21c, d}$$
where
$$\tilde{u}_0(\tilde{r})=1-\tilde{r}^2,\quad \tilde{p}_0(\tilde{s})=-\frac{4}{\mathscr{R}}\tilde{s}. \tag{22a, b}$$

The functional dependence of the perturbative terms on $\theta+\phi$ and $\tilde{r}$ ensures that we are searching for fully developed flow: they do not depend explicitly on $\tilde{s}$, the non-dimensional arc length. We can write, owing to the fact that generally we have
$$\frac{\partial f(\theta+\phi,\tilde{r})}{\partial\tilde{s}}=\frac{\partial f(\theta+\phi,\tilde{r})}{\partial(\theta+\phi)}\frac{\partial(\theta+\phi)}{\partial\tilde{s}}=-\epsilon\lambda\frac{\partial f(\theta+\phi,\tilde{r})}{\partial(\theta+\phi)},$$
the following relations:
$$\frac{\partial\tilde{u}}{\partial\tilde{s}}=-\epsilon^2\lambda\frac{\partial\tilde{u}_1}{\partial(\theta+\phi)},\quad \frac{\partial\tilde{v}}{\partial\tilde{s}}=-\epsilon^2\lambda\frac{\partial\tilde{v}_1}{\partial(\theta+\phi)},\quad \frac{\partial\tilde{w}}{\partial\tilde{s}}=-\epsilon^2\lambda\frac{\partial\tilde{w}_1}{\partial(\theta+\phi)},$$
$$\frac{\partial\tilde{p}}{\partial\tilde{s}}=\frac{d\tilde{p}_0}{d\tilde{s}}-\epsilon^2\lambda\frac{\partial\tilde{p}_1}{\partial(\theta+\phi)}. \tag{23a, b, c, d}$$

If we substitute (21) and (23) in the continuity and momentum equations (17) and (18), we identically satisfy the ϵ^0 equations (Poiseuille flow), and we obtain for the ϵ^1 terms the following set of relations:
$$\frac{\partial\tilde{v}_1}{\partial\tilde{r}}+\frac{1}{\tilde{r}}\frac{\partial\tilde{w}_1}{\partial(\theta+\phi)}+\frac{\tilde{v}_1}{\tilde{r}}=0,$$
$$\tilde{v}_1\frac{d\tilde{u}_0}{d\tilde{r}}=\tilde{r}\sin(\theta+\phi)\frac{d\tilde{p}_0}{d\tilde{s}}+\frac{1}{\mathscr{R}}\left[\left(\frac{\partial}{\partial\tilde{r}}+\frac{1}{\tilde{r}}\right)\left(\frac{\partial\tilde{u}_1}{\partial\tilde{r}}+\tilde{u}_0\sin(\theta+\phi)\right)\right.$$
$$\left.+\frac{1}{\tilde{r}}\frac{\partial}{\partial(\theta+\phi)}\left(\frac{1}{\tilde{r}}\frac{\partial\tilde{u}_1}{\partial(\theta+\phi)}+\tilde{u}_0\cos(\theta+\phi)\right)\right],$$

$$-\tilde{u}_0^2 \sin(\theta+\phi) = -\frac{\partial \tilde{p}_1}{\partial \tilde{r}} - \frac{1}{\mathscr{R}}\frac{1}{\tilde{r}}\frac{\partial}{\partial(\theta+\phi)}\left(\frac{\partial \tilde{w}_1}{\partial \tilde{r}} + \frac{\tilde{w}_1}{\tilde{r}} - \frac{1}{\tilde{r}}\frac{\partial \tilde{v}_1}{\partial(\theta+\phi)}\right),$$

$$-\tilde{u}_0^2 \cos(\theta+\phi) = -\frac{1}{\tilde{r}}\frac{\partial \tilde{p}_1}{\partial(\theta+\phi)} + \frac{1}{\mathscr{R}}\frac{\partial}{\partial \tilde{r}}\left(\frac{\partial \tilde{w}_1}{\partial \tilde{r}} + \frac{\tilde{w}_1}{\tilde{r}} - \frac{1}{\tilde{r}}\frac{\partial \tilde{v}_1}{\partial(\theta+\phi)}\right),$$

where use has been made of the fact that

$$\frac{\partial}{\partial \theta} f(\theta+\phi, \tilde{r}) = \frac{\partial}{\partial(\theta+\phi)} f(\theta+\phi, \tilde{r}).$$

It is easy to recognize that the torsion does not have first-order effects on the motion, because these equations are identical to those of Dean (1927), which can be solved by writing

$$\tilde{u}_1 = \tilde{u}_1^\dagger(\tilde{r})\sin(\theta+\phi), \quad \tilde{v}_1 = \tilde{v}_1^\dagger(\tilde{r})\sin(\theta+\phi),$$

$$\tilde{w}_1 = \tilde{w}_1^\dagger(\tilde{r})\cos(\theta+\phi), \quad \tilde{p}_1 = \tilde{p}_1^\dagger(\tilde{r})\sin(\theta+\phi),$$

and which have as solution the well-known recirculating flow in the plane normal to the axis of the pipe (for a good review of the method, see Ward-Smith 1980). It must be noted finally that this result applies also to the more-general case in which κ = constant and $\tau = \tau(s)$.

4. Conclusions

The introduction of an orthogonal system of coordinates along a spatial curve has allowed us to verify in a simple way that the effect of torsion on a helical pipe flow is a second-order one, while the effect of the curvature is a first-order one. This result disagrees with the results of Wang (1981), who found a first-order effect of torsion on the flow. Probably for higher Reynolds numbers, or for high ratios of torsion to curvature the effects are different, and obviously the consequences of a helix angle are very great for the components of the external forces such as gravity. However, from the point of view of the pure geometrical consequences of the bending and the twisting of the axis, curvature and torsion have not effects of the same order on the pipe flow.

Appendix. Comments on the paper of Wang (1981)

Let us first of all review some notions of differential geometry. If the position of a point is given in terms of the coordinates x^i,

$$\mathbf{P} = \mathbf{P}(x^i),$$

we can define in every point x^i four different systems of base vectors, which give four different representations of the general vector $\mathbf{v}$ (see e.g. Sokolnikoff 1951).

(i) The base vectors are

$$\frac{d\mathbf{P}}{dx^i} = \mathbf{a}_i.$$

They are directed tangentially to the x^i coordinate curve, and the components v^i in the representation

$$\mathbf{v} = v^i \mathbf{a}_i$$

are called the contravariant components of $\mathbf{v}$.

(ii) The base vectors are $\mathbf{a}^i$, such that $\mathbf{a}_i \cdot \mathbf{a}^j = \delta_i^j$. This is the reciprocal base system, and the base vectors are normal to the surface $x^i =$ constant. The components v_i in the representation

$$\mathbf{v} = v_i \mathbf{a}^i$$

are called the covariant components of $\mathbf{v}$.

(iii) The base versors are $\mathbf{b}_i$, and have the same orientation as $\mathbf{a}_i$ but are normalized to unity. The components u^i in the representation

$$\mathbf{v} = u^i \mathbf{b}_i$$

are called the physical contravariant components of $\mathbf{v}$.

(iv) The base versors are $\mathbf{b}^i$, and have the same orientation as $\mathbf{a}^i$, but they also are normalized to unity. The components u_i in the representation

$$\mathbf{v} = u_i \mathbf{b}^i$$

are called the physical covariant components of $\mathbf{v}$.

The fundamental quadratic form, which gives the square of the element of arc, is

$$d\mathbf{P} \,.\, d\mathbf{P} = g_{ij}\, dx^i\, dx^j,$$

where $g_{ij} = \mathbf{a}_i \,.\, \mathbf{a}_j$, $g^{ij} = \mathbf{a}^i \,.\, \mathbf{a}^j$ and we have

$$v_i = g_{ij} v^j, \quad u^j = v^j (g_{jj})^{\frac{1}{2}}, \quad u_i = v_i (g^{ii})^{\frac{1}{2}} = g_{ij} \frac{(g^{ii})^{\frac{1}{2}}}{(g_{jj})^{\frac{1}{2}}} u^j,$$

where it should be noted that the summation convention applied to the repeated indices does not apply to the terms $(g_{jj})^{\frac{1}{2}}$, $(g^{ii})^{\frac{1}{2}}$.

In the Wang system of coordinates $x^1 = r$, $x^2 = \theta$, $x^3 = s$, where the surfaces $s =$ constant are planes normal to a curve $\mathbf{P} = \mathbf{P}(s)$, and r and θ are polar coordinates in these planes. The metric tensor g_{ij} is

$$g^{11} = g_{11} = 1, \quad g_{22} = r^2, \quad g_{33} = (1 - \kappa r \cos\theta)^2 + \tau^2 r^2 = G,$$

$$g^{12} = g_{12} = g_{13} = g^{13} = 0, \quad g_{23} = \tau r^2, \quad g^{23} = -\frac{\tau}{M},$$

$$g^{22} = \frac{G}{r^2 M}, \quad g^{33} = \frac{1}{M}, \quad M = (1 - \kappa r \cos\theta)^2,$$

where $\kappa = \kappa(s)$ and $\tau = \tau(s)$ are the curvature and the torsion of the curve. If v^i are the contravariant components of the velocity $\mathbf{v}$ (in terms of which Wang writes the Navier–Stokes equations), the physical contravariant and covariant components of $\mathbf{v}$ are given by the expressions

$$u^1 = v^1, \quad u^2 = v^2 r, \quad u^3 = v^3 G^{\frac{1}{2}},$$

$$u_1 = u^1, \quad u_2 = \left(\frac{G}{M}\right)^{\frac{1}{2}} u^2 + \frac{\tau r}{M^{\frac{1}{2}}} u^3, \quad u_3 = \frac{\tau r}{M^{\frac{1}{2}}} u^2 + \left(\frac{G}{M}\right)^{\frac{1}{2}} u^3.$$

The description that can be most easily correlated with experiment is the *physical covariant description*, which separates the component u_3 *normal to the plane* $s =$ constant from the component of the velocity *in the plane* $s =$ constant. The description of Wang is in terms of the physical contravariant components (see equation (13) of his paper, where $u = u^1$, $v = u^2$, $w = u^3$ according to our symbolism), which are directed tangentially to the (r, θ, s)-coordinate curves. As a consequence of the fact that his system of coordinates is not orthogonal, the physical contravariant component w, which is parallel to the s-curve, is not normal to the plane $s =$ constant, $(u_3 \neq u^3)$.

If we describe the motion in terms of the physical covariant components, the term dependent on the parameter λ in equation (28) of Wang disappears, and by consequence the effect of the torsion. It is interesting to notice that Wang in his paper has realized that 'since the coordinate system is not orthogonal the velocity w [$= u^3$ in our symbolism], in general, is not perpendicular to the (r, θ)-plane' (see Wang 1981, p. 189), but only with reference to the correct calculation of the flow rate, which must be calculated by recourse to the covariant components.

Obviously in our orthonormal system of coordinates there is no difference between covariant and contravariant components.

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